# Supplementary Material

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**The flatness of bifurcations in 3D dendritic trees: an optimal design.**
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(VP&U, 2012)

Supplementary Material to the section **Probability distribution of the sum of the three bifurcation angles in a random 3D bifurcation:**

First follows the derivation of the probability distribution of the sum of two bifurcation angles in a random 3D bifurcation (Eq (6) in paper VP&U (2012)).

In random bifurcations are the bifurcation angles $\rho, \sigma, \tau$ distributed as $p_1(\rho) = \frac{\sin\rho}{2}$, $p_1(\sigma) = \frac{\sin\sigma}{2}$, and $p_1(\tau) = \frac{\sin\tau}{2}$, respectively. The probability density function $p_2(v)$ of the sum of two bifurcation angles, say $v = \sigma + \tau$, with $\sigma \in [0,\pi]$, $\tau \in [0,\pi]$ and $v \in [0,2\pi]$ , can be obtained by the convolution

$$p_2(v) = \int_{\sigma 1}^{\sigma 2} p_1(\sigma).p_1(v-\sigma).d\sigma \quad . \qquad \text{(S1)}$$

For the primitive function we obtain

$$\int p_1(\sigma).p_1(v-\sigma).d\sigma = \frac{1}{4}\int \sin\sigma.\sin(v-\sigma).d\sigma = \frac{1}{16}(\sin(2\sigma - v) - 2\sigma\cos v) . \qquad \text{(S2)}$$

The term $p_1(\sigma)$ has as domain $\sigma \in [0,\pi]$, and the term $p_1(v-\sigma)$ has as domain $v-\sigma \in [0,\pi]$. The domain of integration is thus restricted by $\sigma \geq 0$, $\sigma \leq \pi$, $v-\sigma \geq 0$, and $v-\sigma \leq \pi$, resulting in $0 \leq \sigma \leq \pi$, and $v-\pi \leq \sigma \leq v$. Thus $\sigma_1 = \max[0, v-\pi]$ and $\sigma_2 = \min[v,\pi]$. The distribution $p_2(v)$ can now be obtained by

$$p_2(v) = \frac{1}{16}\left[\sin(2\sigma - v) - 2\sigma\cos v\right]_{\max[0,v-\pi]}^{\min[v,\pi]} =$$

$$= \frac{1}{16}(\sin(2\min[v,\pi] - v) - 2\min[v,\pi]\cos v - \sin(2\max[0,v-\pi] - v) + 2\max[0,v-\pi]\cos v) =$$

$$= \frac{1}{16}(\sin(2\min[v,\pi] - v) - \sin(2\max[0,v-\pi] - v) - 2\cos v.(\min[v,\pi] - \max[0,v-\pi]))$$

For $v \leq \pi$ we have $\max[0, v-\pi] = 0$ and $\min[v,\pi] = v$

$$p_{2<}(v) = p_2(v \mid v \le \pi) = \frac{1}{16}(\sin v - \sin(-v) - 2\cos v.(v-0)) = \frac{1}{8}(\sin v - v\cos v)$$

For $v > \pi$ we have $\max[0, v-\pi] = v-\pi$ and $\min[v,\pi] = \pi$

$$p_{2>}(v) = p_2(v \mid v > \pi) = \frac{1}{16}\left(\sin(2\pi - v) - \sin(2v - 2\pi - v) - 2\cos v.(\pi - v + \pi)\right) =$$

$$= \frac{1}{16}\left(2\sin(2\pi - v) - 2(2\pi - v)\cos v\right) = -\frac{1}{8}\left(\sin v + (2\pi - v)\cos v\right)$$

Thus, in summary we obtain the expressions of Eq. (6) in paper VP&U (2012):

$$\begin{aligned} p_2(v \mid v \le \pi) &= \frac{1}{8}(\sin v - v\cos v) \\ p_2(v \mid v > \pi) &= -\frac{1}{8}\left(\sin v + (2\pi - v)\cos v\right) \end{aligned} \qquad \text{(S3)}$$

It is easy verified that for positive *x*, $p_2(\pi - x) = p_2(\pi + x)$, showing the symmetry of the distribution $p_2(v)$ around angle $v = \pi$.

**Probability distribution of the sum of the three bifurcation angles in a random 3D bifurcation**

For the sum *S* of the three bifurcation angles we have $S = \rho + \sigma + \tau = \rho + v$, with $\rho \in [0,\pi]$, $\sigma \in [0,\pi]$, $\tau \in [0,\pi]$, $v \in [0,2\pi]$, and $S \in [0,2\pi]$. Although the random bifurcation is composed of three independent random vectors in 3D space, the three bifurcation angles are not independent from each other. While two of the three bifurcation angles, say $\sigma$ and $\tau$, can still be chosen independently and randomly according to $p_1(\sigma)$ and $p_1(\tau)$, respectively, the third follows another distribution. With the selection of $\sigma$ and $\tau$, two planes are formed (of the parent segment and each of the daughter segments), but the angle between these planes (i.e., dihedral angle $\lambda$, see Figure 2H in VP&U (2012)) can still be chosen independently and randomly according to a uniform distribution $p(\lambda \mid 0 \le \lambda \le \pi)$. The probability density function $p_3(S)$ of the sum of the three bifurcation angles $S = \rho + \sigma + \tau = \rho + v$ can subsequently be expressed by the convolution

$$p_3(S) = p_3(S = \rho + \sigma + \tau) = p_3(S = \rho + v) = \int_0^S \left( \int_0^v p_1(x).p_1(v - x) p_\rho(S - v) dx \right) dv \qquad \text{(S4)}$$

with

$$p_1(x) = \frac{\sin x}{2},\ p_1(v - x) = \frac{\sin(v - x)}{2},\ p(\lambda \mid 0 \le \lambda \le \pi) = \frac{1}{\pi}$$

The probability density function $p_\rho(\rho)$ of angle $\rho$ can be obtained by expressing angle $\rho$ as a function of $\lambda$ and using the probability density function $p(\lambda)$ of the dihedral angle $\lambda$. The dihedral angle $\lambda$ and bifurcation angle ρ are related via (Figure 2H in VP&U (2012))

$$\cos\lambda = \frac{\cos\rho - \cos\sigma\cos\tau}{\sin\sigma\sin\tau} \text{ or } \cos\rho = \cos\sigma\cos\tau + \sin\sigma\sin\tau\cos\lambda$$

To relate the probability density functions $p(\rho)$ and $p(\lambda)$ we use the equality

$$\int_{\rho_1}^{\rho_2} p(\rho)d\rho = \Pr(\rho_1 \le \rho \le \rho_2) = \Pr(\lambda_1 \le \lambda \le \lambda_2) = \int_{\lambda_1}^{\lambda_2} p(\lambda)d\lambda \quad . \tag{S5}$$

The dihedral angle $\lambda$ is uniformly distributed over $\lambda \in [0,\pi]$ with $p(\lambda) = \frac{1}{\pi}$ (Eq. (16) in VP&U (2012)) and we obtain

$$\int_{\rho_1}^{\rho_2} p(\rho)d\rho = \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} d\lambda$$

$$\frac{d\rho}{d\lambda} = \frac{d\rho}{d\cos\rho}.\frac{d\cos\rho}{d\lambda} = -\frac{1}{\sin\rho}.\frac{d\cos\rho}{d\lambda} = \frac{\sin\sigma\sin\tau\sin\lambda}{\sin\rho}$$

and thus

$$d\lambda = \frac{\sin\rho}{\sin\sigma\sin\tau\sin\lambda}d\rho \tag{S6}$$

In order to express $\sin\lambda$ in terms of angle $\rho$ we follow with

$$\sin\lambda = \sqrt{1-\cos^2\lambda} = \sqrt{1-\left(\frac{\cos\rho - \cos\sigma\cos\tau}{\sin\sigma\sin\tau}\right)^2} = \ldots =$$

$$= \frac{\sqrt{-\cos(\sigma-\tau)\cos(\sigma+\tau) - \cos^2\rho + 2\cos\rho\cos\sigma\cos\tau}}{\sin\sigma\sin\tau} .$$

Thus

$$d\lambda = \frac{\sin\rho}{\sqrt{-\cos(\sigma+\tau)\cos(\sigma-\tau) - \cos^2\rho + 2\cos\rho\cos\sigma\cos\tau}}d\rho \tag{S7}$$

and

$$\int_{\rho_1}^{\rho_2} p(\rho)d\rho = \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} d\lambda = \frac{1}{\pi}\int_{\rho_1}^{\rho_2} \frac{\sin\rho}{\sqrt{-\cos(\sigma-\tau)\cos(\sigma+\tau) - \cos^2\rho + 2\cos\rho\cos\sigma\cos\tau)}}d\rho .$$

For the probability density function $p_\rho(\rho)$ we finally obtain

$$p_\rho(\rho)=\frac{\sin\rho}{\pi\sqrt{-\cos(\sigma-\tau)\cos(\sigma+\tau)-\cos^2\rho+2\cos\rho\cos\sigma\cos\tau}} \qquad \text{(S8)}$$

with

$$p_\rho(S-v)=\frac{\sin(S-v)}{\pi\sqrt{-\cos(\sigma-\tau)\cos(\sigma+\tau)-\cos^2(S-v)+2\cos(S-v)\cos\sigma\cos\tau}}=$$

$$=\frac{\sin(S-v)}{\pi\sqrt{-\cos(2x-v)\cos v-\cos^2(S-v)+2\cos(S-v)\cos x\cos(v-x)}}$$

Eq. (S4) can now be rewritten as

$$p_3(S)=\int_0^S\left(\int_0^v p_1(x).p_1(v-x)p_\rho(S-v)dx\right)dv=$$

$$=\frac{1}{4\pi}\int_0^S\left(\int_0^s\frac{\sin x\sin(v-x)\sin(S-v)}{\sqrt{-\cos(2x-v)\cos v-\cos^2(S-v)+2\cos(S-v)\cos x\cos(v-x)}}dx\right)dv=$$

$$=\frac{1}{4\pi}\int_0^S\sin(S-v)\left(\int_0^v\frac{\sin x\sin(v-x)}{\sqrt{-\cos(2x-v)\cos v-\cos^2(S-v)+2\cos(S-v)\cos x\cos(v-x)}}dx\right)dv \qquad \text{(S9)}$$

which proves Eq (7) in VP&U (2012). Further attempts to reduce this equation were, unfortunately, not successful. Rather than solving this expression numerically, the distribution for the bifurcation angle sum has been obtained by generating 1000000 random bifurcations as described in Section DISTRIBUTIONS OF MEASURES OF FLATNESS OF RANDOM 3D BIFURCATIONS of VP&U (2012), and calculating the sum of the three bifurcation angles.

Supplementary Material to

**APPENDIX - SOLID ANGLE OF A TRIANGULAR PYRAMID FORMED BY A 3D BIFURCATION**

A 3D bifurcation defines a triangular pyramid with the bifurcation point as apex and the tips of its branches defining the planar base triangle. By definition the solid angle of a triangular pyramid equals (Figure A4)

$$\Omega_P = \frac{\Delta GEF}{l^2} \tag{A16}$$

with $\Delta EFG$ equal to the surface area of the spherical triangle on the sphere through the tips and centered at the bifurcation point and $l$ the radius of the sphere. The surface area of the spherical triangle $\Delta EFG$ is equal to $\Delta EFG = l^2\varepsilon$ (e.g., Polyanin and Manzhirov, 2007) and thus

$$\Omega_P = \varepsilon \tag{A17}$$

with $\varepsilon$ the spherical excess of the spherical triangle *EFG*. For a spherical triangle with sides *e*, *f*, *g* and angles $\gamma$, $\mu$, $\nu$ (Figure A4), we have the following relations (e.g., Polyanin and Manzhirov, 2007)

$0 < e + f + g < 2l\pi$, $\pi < \gamma + \mu + \upsilon < 3\pi$, and the spherical excess $\varepsilon = \gamma + \mu + \upsilon - \pi$.

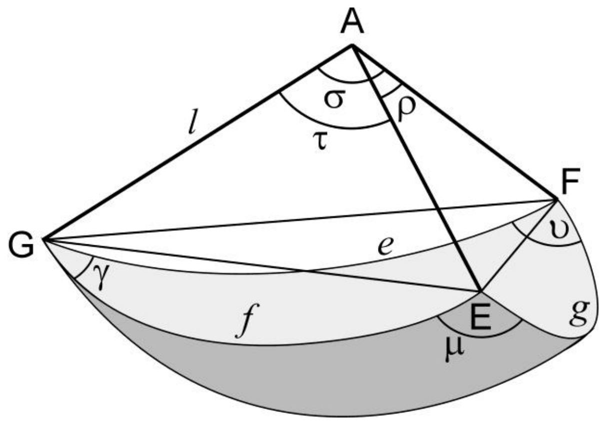


**Figure A4.** The solid angle of a triangular pyramid formed by equal segments parts $\ell$ from the node A, of a 3D bifurcation subtends a spherical triangle GEF with spherical

center at the apex of pyramid and radius $\ell$. The sides of the spherical triangle are denoted by *e*, *f* and *g*, and its spherical angles by $\gamma$, $\mu$ and $\nu$.

Casey (1889, reprinted 2007) has reported an expression for $\varepsilon / 2$ for the unit sphere ($l = 1$), in his Eq. 359 on p.87, i.e.,

$$\cos\frac{\varepsilon}{2} = \frac{1 + \cos e + \cos f + \cos g}{4\cos\frac{e}{2}\cos\frac{f}{2}\cos\frac{g}{2}} \tag{A18}$$

called the Euler's Formula for one-half of the spherical excess expressed in terms of the sides of the spherical triangle (Donnay, 1945, reprinted 2009). The angles $\tau$, $\rho$ and $\sigma$ relate to the unit circle segments *f*, *g* and *e*, respectively, as

$$f = \tau\,,\; g = \rho\,, \text{and}\;\; e = \sigma\;. \tag{A19}$$

This leads to a simple equation for $\Omega_P$ in terms of the 3 bifurcation angles

$$\Omega_P = 2\arccos\left[\frac{1+\cos\rho+\cos\sigma+\cos\tau}{4\cos\frac{\rho}{2}\cos\frac{\sigma}{2}\cos\frac{\tau}{2}}\right] \tag{A20}$$

which shows the symmetry in $\rho$, $\sigma$, and $\tau$ in $\Omega_P$. Eq. (A20) is undefined, when one of the 3 bifurcation angles is $\pi$, in such a condition the bifurcation is planar and thus $\Omega_P = 2\pi$.

For the derivation of Euler's Formula (A18) in VP&U (2012) , Donnay (1945, reprinted 2011) applied the stereographic projections method in combination with Cesàro's method of 'Triangles of Elements'. For reason of convenience and completeness a derivation of Euler's Formula based on the approach of Donnay (1945) will be given below.

**Supplementary Material for the derivation of Euler's Formula based on the approach of Donnay (1945)**

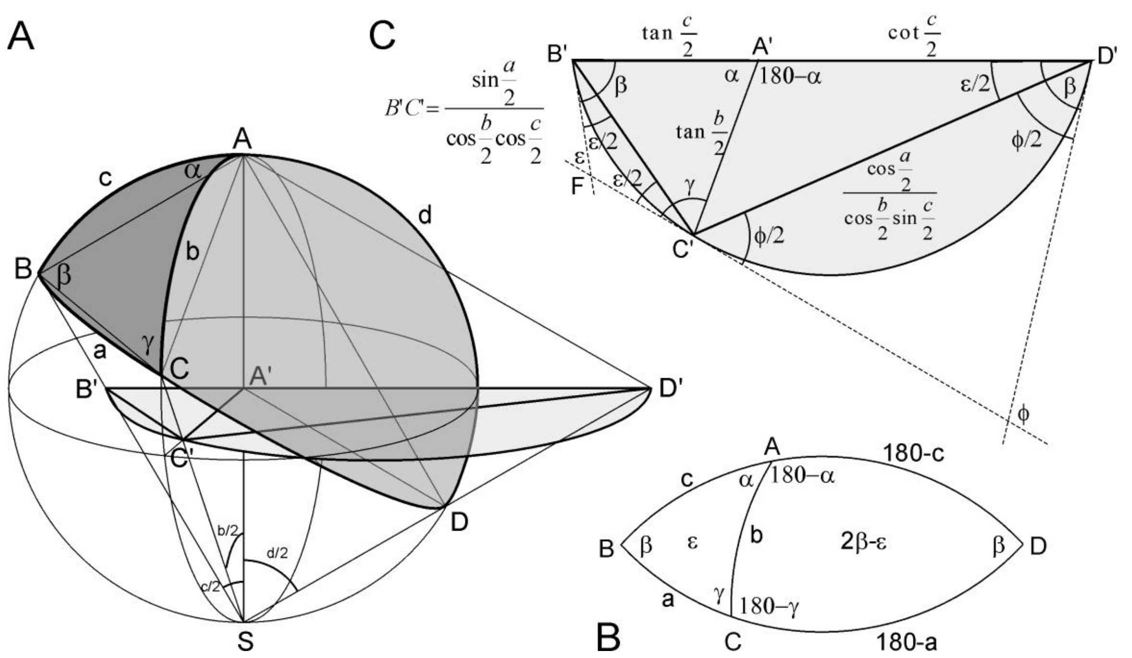


**Figure S1. (A)** Stereographic projection of spherical triangle ABC and its complementary spherical triangle ADC onto equatorial plane with the South pole as projection point. **(B)** Sides and angles of the lune ABCD. **(C)** Sides and angles of the planar triangles in the equatorial projection plane.

**Spherical excess of a spherical triangle**

The spherical triangle ABC in Figure S1A is composed of the great circles through the vertices A, B and C of the unit sphere Without loss of generality vertex A is positioned at the North pole of the unit sphere with side AB in the plane of drawing. Continuing the great circle segments BA and BC they will meet in vertex D at the opposite side of B with respect to the center of the sphere. The spherical surface bounded by two great semi-circles BAD and BCD is called a lune (Figure S1B). The angle at vertex D equals angle $\beta$ at the opposite vertex B. Because the sides BAD and BCD comprise semi-circles, the length of AD equals $d = 180 - c$ and the length of CD is equal to $180 - a$.

**Angles of the plane triangles A'B'C' and A'D'C'**

The planar triangles A'B'C' and A'D'C' in Figure S1A are the stereographical projections of the spherical triangle ABC and its complementary spherical triangle ADC, respectively, onto the equatorial plane with the South pole as projection point. The angles $\alpha$, $\beta$, and $\gamma$ of the spherical triangle ABC are preserved in this projection (Figure S1A,C). The angles $\beta$ and $\gamma$ of the spherical triangle exceed those of the plane triangle $\angle A'B'C'$ and $\angle A'C'B'$, respectively, each with half of the spherical excess $\varepsilon/2$ (Figure S1C). With the angle sum of the quadrangle A'B'FC' equal to 360, $\alpha + \beta + \gamma + (180 - \varepsilon) = 360$, we obtain

$$\varepsilon = \alpha + \beta + \gamma - 180 . \qquad \text{(S10)}$$

Because the stereographic projection of a circle is again a circle (Casey, 1889), the angles $\angle A'B'C'$ and $\angle A'C'B'$ of the plane triangle can be expressed as $\angle A'B'C' = \beta - \varepsilon/2$, and $\angle A'C'B' = \gamma - \varepsilon/2$. For the spherical excess $\phi$ of the spherical triangle A'C'D' we have in a similar sense $180 - \alpha + 180 - \gamma + \beta + 180 - \varphi = 360$ or $\varphi = -\alpha - \gamma + \beta + 180 = 2\beta - \varepsilon$. Thus, $\varphi/2 = \beta - \varepsilon/2$, and thus $\angle A'D'C' = \varepsilon/2$, implicating that the spherical excess ε can be obtained by solving triangle *A'D'C'*.

**Sides of the plane triangle A'D'C'**

As triangles SAC and SC'A' in Figure S1A share the common angle *b/2* and have both right angles $\angle SCA$ and $\angle SA'C'$, they are congruent, and thus $\frac{SC'}{SA'} = \frac{SA}{SC}$ or $SA.SA' = SC.SC'$. Similarly, $SA.SA' = SD.SD'$. Thus $SC.SC' = SD.SD'$, making the triangles SDC and SC'D' also congruent. This gives us for the sides

$$\frac{D'C'}{DC} = \frac{SD'}{SC} \text{ and } D'C' = DC\frac{SD'}{SC}. \qquad \text{(S11)}$$

Angle $\angle CA'D$ equals side 180-*a* of the spherical triangle for the unit sphere, and thus we obtain the length DC

$$DC = 2\sin\frac{180 - a}{2} = 2\cos\frac{a}{2}. \qquad \text{(S12)}$$

The length of SD' can be obtained via

$$\cos\frac{d}{2}=\frac{A'S}{SD'}=\frac{1}{SD'}\text{, and thus } SD'=\frac{1}{\cos\frac{d}{2}}=\frac{1}{\cos\frac{180-c}{2}}=\frac{1}{\sin\frac{c}{2}}. \quad \text{(S13)}$$

Further we have $SC=2\cos\frac{b}{2}$ . (S14)

Inserting (S12) - (S14) into (S11) we obtain for *D'C'*

$$D'C'=\frac{\cos\frac{a}{2}}{\cos\frac{b}{2}\sin\frac{c}{2}}. \quad \text{(S15)}$$

The lengths of sides *A'C'* and *A'D'* can be obtained directly via $A'C'=\tan\frac{b}{2}$ and $A'D'=\tan\frac{d}{2}=\tan\frac{180-c}{2}=\cot\frac{c}{2}$. Having in the plane triangle A'C'D' half of the spherical excess ε/2 as one of the corner angles ($\angle A'D'C'=\varepsilon/2$) its value can now be obtained using the cosine rule

$$A'C'^2=A'D'^2+C'D'^2-2A'D'.C'D'.\cos\frac{\varepsilon}{2}. \quad \text{(S16)}$$

Thus:

$$\cos\frac{\varepsilon}{2}=\frac{A'D'^2+C'D'^2-A'C'^2}{2.A'D'.C'D'}=\frac{\cot^2\frac{c}{2}+\cos^2\frac{a}{2}/\left(\cos^2\frac{b}{2}\sin^2\frac{c}{2}\right)-\tan^2\frac{b}{2}}{2\cot\frac{c}{2}\cos\frac{a}{2}/\left(\cos\frac{b}{2}\sin\frac{c}{2}\right)}=\ldots=$$

$$=\frac{\cos^2\frac{a}{2}+\cos^2\frac{b}{2}+\cos^2\frac{c}{2}-1}{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} \quad \text{(S17)}$$

which proves (A18).

**Sides of triangle A'B'C'**

The sides of triangle A'B'C' can be obtained in a similar way with

$$B'C'=\frac{\sin\frac{a}{2}}{\cos\frac{b}{2}\cos\frac{c}{2}},\quad A'B'=\tan\frac{c}{2}\text{ and } A'C'=\tan\frac{b}{2}.$$

Multiplying the sides of triangle A'B'C' with the term $\cos\frac{b}{2}\cos\frac{c}{2}$ gives Cesàro's triangle of elements relative to angle α (Figure S2A). Multiplying the sides of triangle A'D'C' with the term $\cos\frac{b}{2}\sin\frac{c}{2}$ gives Cesàro's triangle of elements relative to the angle 180-$\alpha$ (Figure S2B), also called the derived triangle relative to angle $\alpha$ (Donnay, 1945).

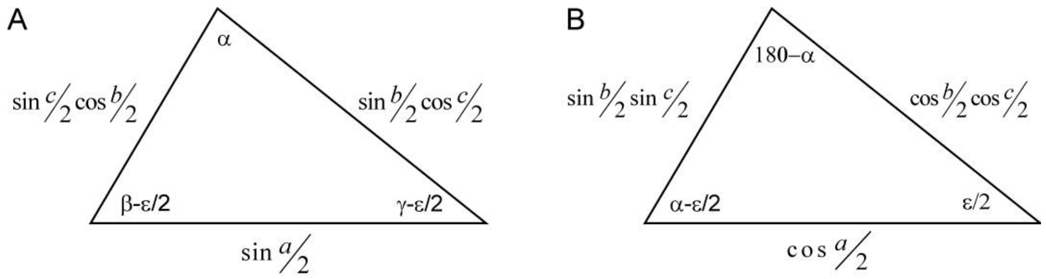


**Figure S2. (A)** Cesaro's triangle of elements relative to angle $\alpha$ and **(B)** derived triangle relative to angle $\alpha$.

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